1A: Vectors and Matrices - Vectors, determinants and planes

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<2019-04-18 Thu>

Contents

1 1A: Vectors and Matrices - Vectors, determinants and planes

Vectors are basic to this course. We will learn to manipulate them algebraically and geometrically. They will help us simplify the statements of problems and theorems and to find solutions and proofs.

Determinants measure volumes and areas. They will also be important in part B when we use matrices to solve systems of equations.

Last Modification: **2020-09-16 Wed 14:01**

Captured On *[2020-01-18 Sat]*

Source Part A: Vectors, Determinants and Planes | 1. Vectors and Matrices | Multivariable Calculus | Mathematics | MIT OpenCourseWare

2 Session 1: Vectors

Captured On *[2020-02-05 Wed 19:40]*

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2.1 Chalkboard

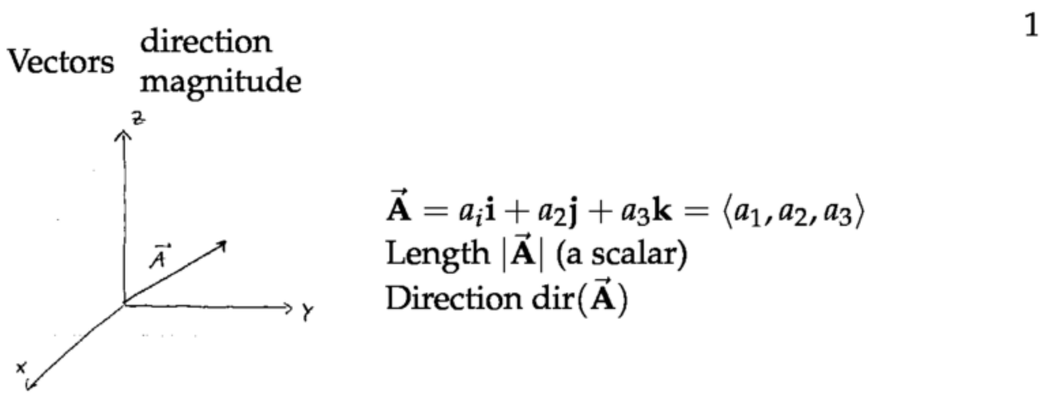


Figure 1: Definition of vectors

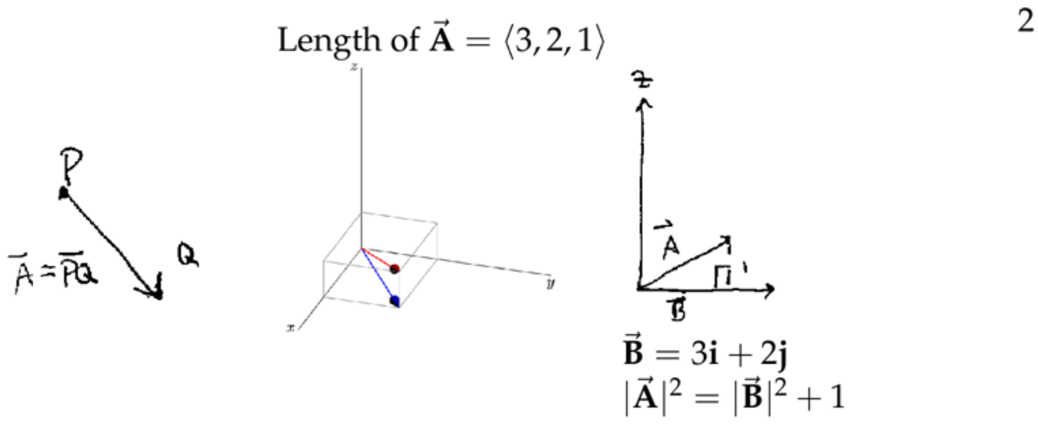


Figure 2: Length of a vector

3

$$|\vec{\mathbf{B}}| = \sqrt{3^2 + 2^2} = \sqrt{13}$$
$$|\vec{\mathbf{A}}| = \sqrt{|\vec{\mathbf{B}}|^2 + 1} = \sqrt{14}$$

In general, $\vec{\mathbf{A}} = \langle a_1, a_2, a_3 \rangle$

$$|\vec{\mathbf{A}}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Addition: $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

Figure 3: Modules and addition

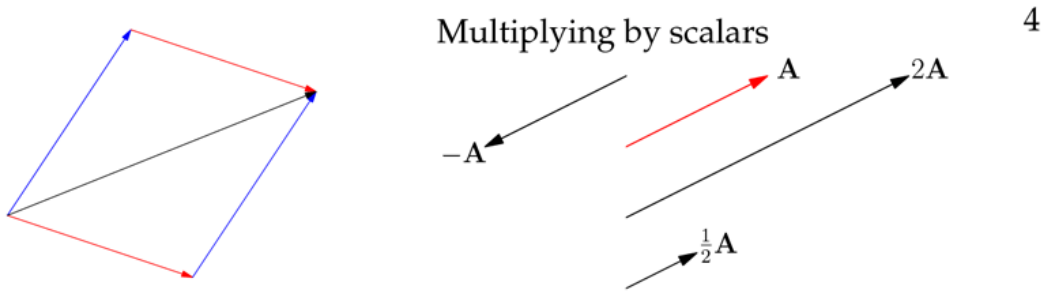


Figure 4: Multiplying by scalars

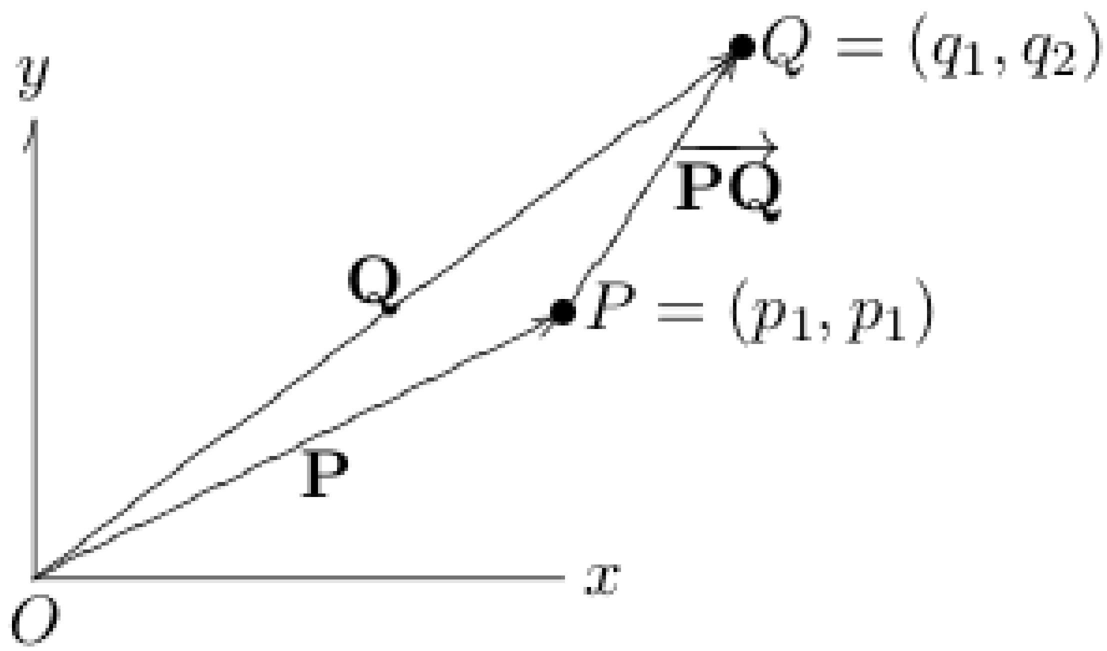
2.2 Which is the vector between 2 points?

2.2.1 Front

For two points P and Q

2.2.2 Back

- $\mathbf{PQ} = \mathbf{Q} - \mathbf{P}$
- $\mathbf{PQ}$ is the displacement from P to Q



2.3 Which is the notation for vectors and points?

2.3.1 Front

How we can write them?

2.3.2 Back

- Points: (a_1, a_2)
- Vectors:
  - $< a_1, a_2 >= a_1\mathbf{i} + a_2\mathbf{j}$
  - $\mathbf{P} = \mathbf{OP}$ is the vector from the origin to P
- A real number is a *scalar*

2.4 Which is the magnitude of a vector in 3D?

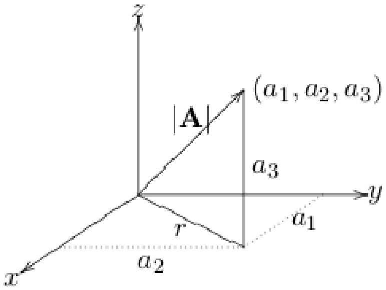
2.4.1 Front

$|\langle a_1, a_2, a_3 \rangle|$

2.4.2 Back

$$|a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}| = |\langle a_1, a_2, a_3 \rangle| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

You can see this in the figure below, where $r = \sqrt{a_1^2 + a_2^2}$ and $|\mathbf{A}| = \sqrt{r^2 + a_3^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$.



2.5 Which is a unit vector?

2.5.1 Front

2.5.2 Back

Is any vector with unit length $\hat{\mathbf{u}}$

- $|\hat{\mathbf{u}}| = 1$
- Special vectors
  - $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$
  - $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$
  - $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$

2.6 How can we get a vector of the median of the triangle?

2.6.1 Front

Triangle: ABC, from vertex: A

2.6.2 Back

$\text{AM} = \frac{1}{2}(\vec{B} + \vec{C}) - \mathbf{A}$

2.7 How we can find the unit vector from any vector?

2.7.1 Front

2.7.2 Back

$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$

3 Session 2: Dot Products

Captured On *[2020-02-05 Wed 19:41]*

Source Session 2: Dot Products | Part A: Vectors, Determinants and Planes | 1. Vectors and Matrices | Multivariable Calculus | Mathematics | MIT OpenCourseWare

3.1 Chalkboard

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DOT PRODUCT

Definition: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$

This is a scalar.

Geometrically $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |\vec{\mathbf{A}}||\vec{\mathbf{B}}| \cos(\theta)$

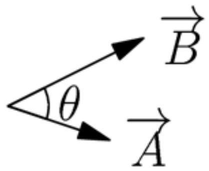

Figure 5: Dot Product Definition

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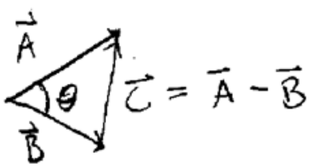
What does geometric definition mean?

1) $\vec{\mathbf{A}} \cdot \vec{\mathbf{A}} = |\vec{\mathbf{A}}|^2 \cos(\theta) = |\vec{\mathbf{A}}|^2 = a_1^2 + a_2^2 + a_3^2$ ✓



Law of cosines

$|\vec{\mathbf{c}}|^2 = |\vec{\mathbf{A}}|^2 + |\vec{\mathbf{B}}|^2 - 2|\vec{\mathbf{A}}||\vec{\mathbf{B}}| \cos(\theta)$

Figure 6: What does geometic definition mean?

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$$
\begin{aligned}
|\vec{\mathbf{C}}|^2 &= \vec{\mathbf{C}} \cdot \vec{\mathbf{C}} = (\vec{\mathbf{A}} - \vec{\mathbf{B}}) \cdot (\vec{\mathbf{A}} - \vec{\mathbf{B}}) \\
&= \vec{\mathbf{A}} \cdot \vec{\mathbf{A}} - \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} - \vec{\mathbf{B}} \cdot \vec{\mathbf{A}} + \vec{\mathbf{B}} \cdot \vec{\mathbf{B}} \\
&= |\vec{\mathbf{A}}|^2 + |\vec{\mathbf{B}}|^2 - \underbrace{2\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}
\end{aligned}
$$

Figure 7: Dot product of combined vectors

3.2 What is the dot product of 2 vectors?

3.2.1 Front

3.2.2 Back

- Is one way to combining ("multiplying") two vectors
- The output is a scalar
- Algebraically
  - $A \cdot B = a_1 b_1 + a_2 b_2$
- Geometrically
  - $A \cdot B = |A||B| \cos(\theta)$

https://i.imgur.com/FLqXbRI.png

3.3 How can we proof the dot product geometrically?

3.3.1 Front

3.3.2 Back

https://i.imgur.com/FLqXbRI.png

- Law of cosines
  - $|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos(\theta)$
- Expanding $|A - B|^2$
  - $|\mathbf{A} - \mathbf{B}|^2 = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2\mathbf{A} \cdot \mathbf{B}$
- Comparing the 2 equations
  - $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos(\theta)$

3.4 What does mean that $A \cdot B = 0$?

3.4.1 Front

3.4.2 Back

Two vectors are perpendicular to each other, we say they are *orthogonal*

- $\cos(\pi/2) = 0$
- $\mathbf{A} \perp \mathbf{B} \Leftrightarrow \mathbf{A} \cdot \mathbf{B} = 0$

3.5 Which is the dot product of the 1 vector by itself?

3.5.1 Front

$\mathbf{A} \cdot \mathbf{A}$

3.5.2 Back

- Algebraically
  - $\mathbf{A} \cdot \mathbf{A} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = |\mathbf{A}|^2$
- Geometrically
  - $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{A}| \cos\theta = |\mathbf{A}|^2$

4 Session 3: Uses of the Dot Product: Lengths and Angles

Captured On *[2020-02-05 Wed 19:43]*

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4.1 Chalkboard

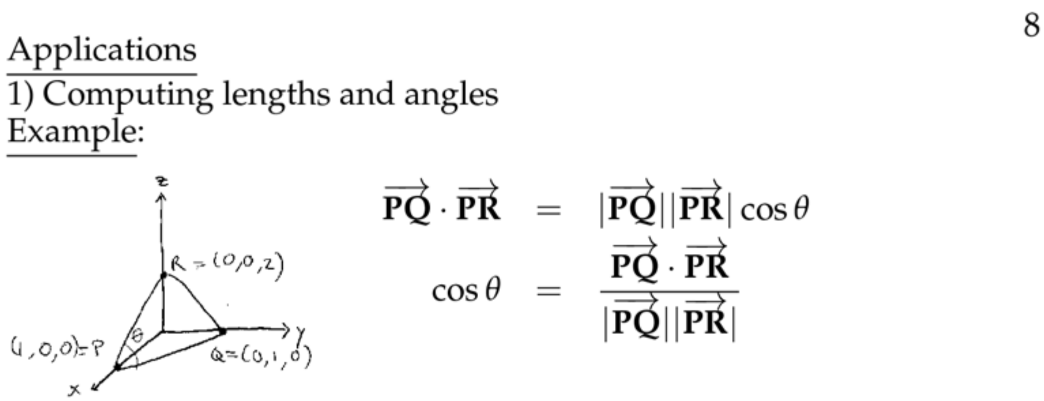

Figure 8: Computing lengths and angles

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$$\cos\theta = \frac{\overrightarrow{\mathbf{PQ}} \cdot \overrightarrow{\mathbf{PR}}}{|\overrightarrow{\mathbf{PQ}}||\overrightarrow{\mathbf{PR}}|} = \frac{\langle -1,1,0\rangle \cdot \langle -1,0,2\rangle}{\sqrt{(-1)^2+1^2+0^2}\sqrt{(-1)^2+0^2+2^2}}$$

$$= \frac{1+0+0}{\sqrt{2}\cdot\sqrt{5}} = \frac{1}{\sqrt{10}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \approx 71.5^\circ$$

Figure 9: Resolution of computing an angle

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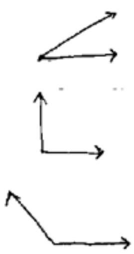
Sign of $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ > 0 if $\theta < 90^\circ$

$= 0$ if $\theta = 90^\circ$

< 0 if $\theta > 90^\circ$

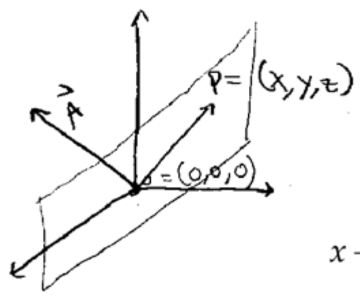


2) Detect Orthogonality

Figure 10: Meaning of sign of a dot product

11

Example: $x + 2y + 3z = 0$ is equation of a plane



$$\overrightarrow{\mathbf{OP}} = \langle x,y,z\rangle$$
$$\vec{\mathbf{A}} = \langle 1,2,3\rangle$$

$$x + 2y + 3z = 0 \Leftrightarrow \overrightarrow{\mathbf{OP}} \cdot \vec{\mathbf{A}} = 0 \Leftrightarrow \overrightarrow{\mathbf{OP}} \perp \vec{\mathbf{A}}$$

Figure 11: Detect orthogonality

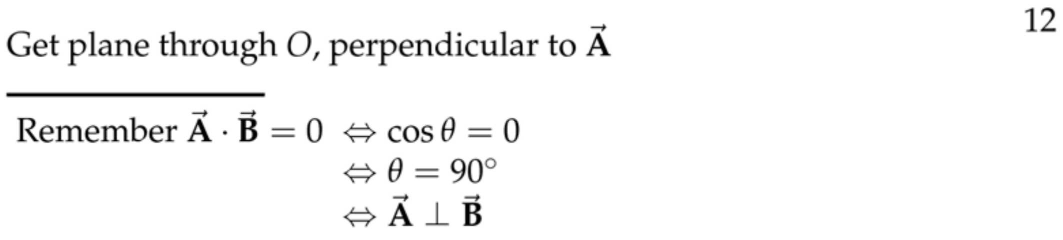


Figure 12: Plane throught O, perpendicular to $\vec{A}$

5 Session 4: Vector Components

Captured On *[2020-02-05 Wed 19:46]*

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5.1 Chalkboard

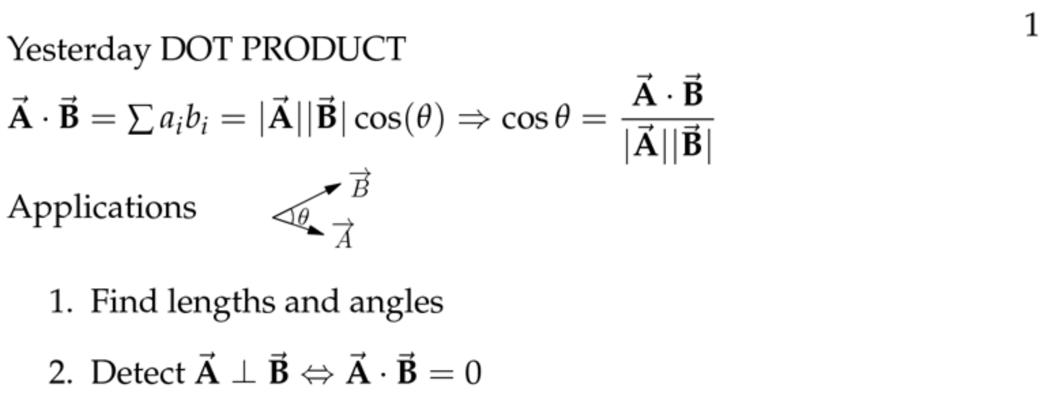


Figure 13: Review of dot product

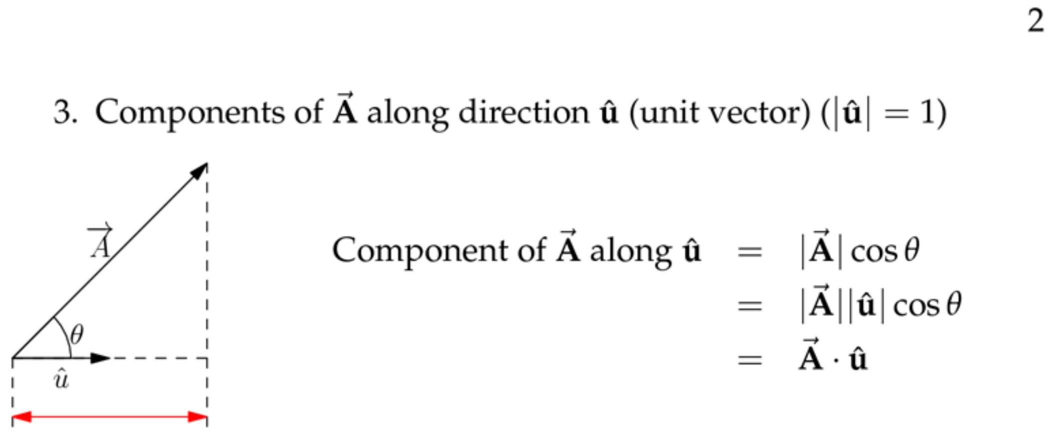


Figure 14: Components of $\vec{A}$ along direction $\vec{u}$

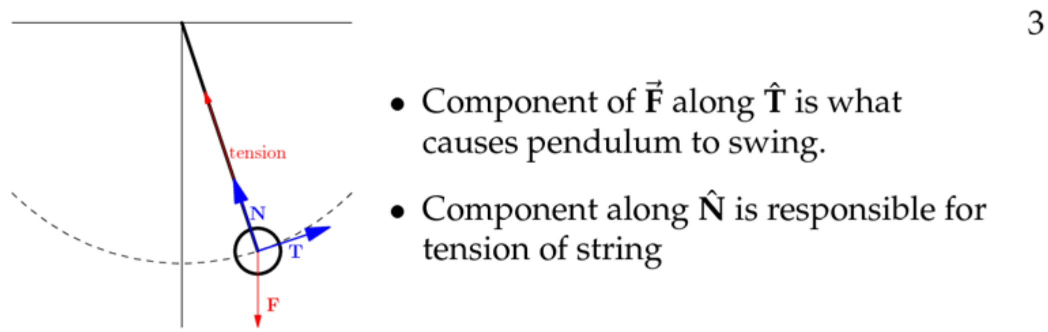


Figure 15: Pendulum problem with projections

5.2 Which is it the angle between 2 vectors?

5.2.1 Front

5.2.2 Back

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$$

5.3 Which is the vector rotated $90°$ counter-clockwise?

5.3.1 Front

$\mathbf{A} = \langle a_1, a_2 \rangle$

5.3.2 Back

$\langle -a_2, a_1 \rangle$

https://i.imgur.com/6DbrmuZ.png

5.4 How we can know which kind of angle there is between 2 vectors?

5.4.1 Front

- Acute, right or obtuse

5.4.2 Back

- $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$
- Numerator can say if there is positive or negative, if it's negative it's obtuse because of $\cos(\theta) < 0$ when $\theta > \pi/2$.
- We will need to check if $\mathbf{A}\cdot\mathbf{B} = 0$, $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|$

5.5 Which is the components of A along direction û?

5.5.1 Front

5.5.2 Back

- $|\hat{\mathbf{u}}| = 1$

https://i.imgur.com/DT9CJps.png

6 Session 5: Area and Determinants in 2D

Captured On *[2020-02-05 Wed 19:47]*

Source Session 5: Area and Determinants in 2D | Part A: Vectors, Determinants and Planes | 1. Vectors and Matrices | Multivariable Calculus | Mathematics | MIT OpenCourseWare

6.1 Chalkboard

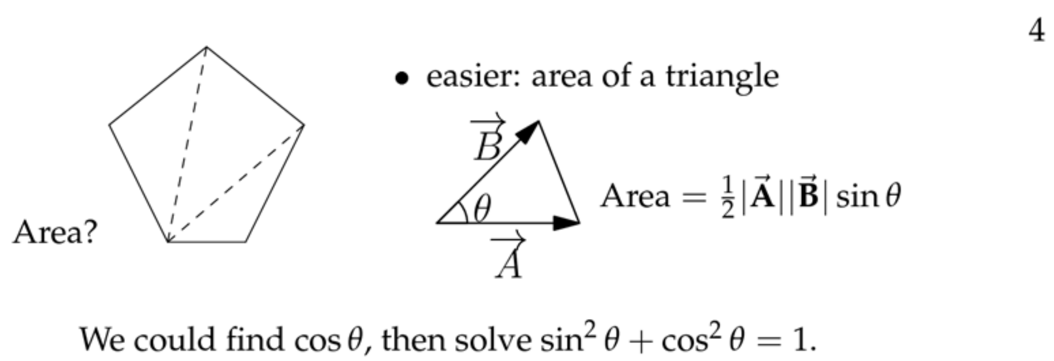
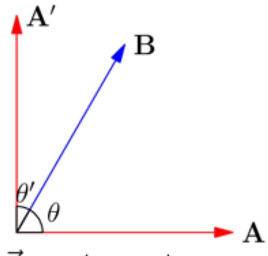


Figure 16: Area within 2 vectors

5



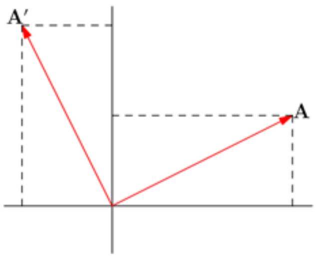
$\vec{\mathbf{A}} = \langle a_1, a_2 \rangle$

$$\vec{\mathbf{A}}' = \vec{\mathbf{A}} \text{ rotated } 90^\circ$$
$$\theta' = \tfrac{\pi}{2} - \theta$$
$$\cos(\theta') = \sin(\theta)$$

$$\begin{aligned} &|\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin(\theta) \\ &= |\vec{\mathbf{A}}'||\vec{\mathbf{B}}|\cos(\theta) \\ &= \vec{\mathbf{A}}' \cdot \vec{\mathbf{B}} \\ &= \langle -a_2, a_1 \rangle \cdot \langle b_1, b_2 \rangle \\ &= a_1 b_2 - a_2 b_1 \end{aligned}$$

Figure 17: Rotated vector $\vec{A}$

6



$\vec{\mathbf{A}}' = \langle -a_2, a_1 \rangle$

$$= \det(\vec{\mathbf{A}}, \vec{\mathbf{B}})$$
$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

determinant of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$

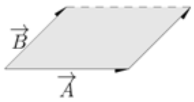
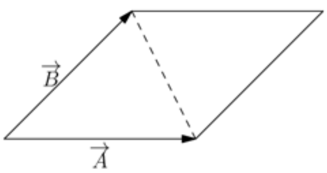
= ±area of 

Figure 18: Area as determinant of 2 vectors

7



$$\begin{aligned} \pm\text{area}(\Diamond) &= |\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin\theta \\ &= \det(\vec{\mathbf{A}}, \vec{\mathbf{B}}) \\ \pm\text{area}(\triangle) &= \tfrac{1}{2}|\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin\theta \\ &= \tfrac{1}{2}\det(\vec{\mathbf{A}}, \vec{\mathbf{B}}) \end{aligned}$$

Area  = abs. value of det.

Figure 19: Area of parallelogram and triangle

6.2 What is the area between 2 vectors?

6.2.1 Front

6.2.2 Back

https://i.imgur.com/neOoGgP.png

- Length and determinant
  - $\mathbf{A}' = \mathbf{A}$ rotated $90°$
  - $\theta' = \pi/2 - \theta$
  - $\cos\theta' = \sin\theta$
  - Area of parallelogram
    - $|\mathbf{A}||\mathbf{B}|\sin\theta = |\mathbf{A}'||\mathbf{B}|\cos\theta' = \mathbf{A}' \cdot \mathbf{B} = \langle -a_2, a_1\rangle \cdot \langle b_1, b_2\rangle = a_1b_2 - a_2b_1 = \det(\mathbf{A}, \mathbf{B})$
    - absolute value of determinant of $\mathbf{A}$ and $\mathbf{B}$
  - Area of triangle
    - $\left|\frac{1}{2}\det(\mathbf{A}, \mathbf{B})\right|$
- Cross product
  - Area of parallelogram
    - $|\mathbf{A} \times \mathbf{B}|$
  - Area of triangle
    - $\frac{1}{2}|\mathbf{A}| \times |\mathbf{B}|$

7 Session 6: Volumes and Determinants in Space

Captured On *[2020-02-05 Wed 19:49]*

Source Session 6: Volumes and Determinants in Space | Part A: Vectors, Determinants and Planes | 1. Vectors and Matrices | Multivariable Calculus | Mathematics | MIT OpenCourseWare

7.1 Chalkboard

Determinant in space. 3 vectors $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}$ 8

$$\det(\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Figure 20: Determinant in space

Theorem: Geometrically, 9
$\det(\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}) = \pm$volume of parallelepiped

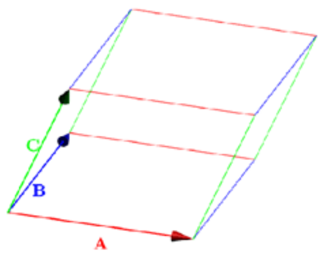


Figure 21: Volume of parallelepiped

7.2 What is the volume between 3 vectors?

7.2.1 Front

7.2.2 Back

- With determinants
  - $|\det(\mathbf{A}, \mathbf{B}, \mathbf{C})|$
- With cross product
  - Volume = area(base) height
  - $V = |\mathbf{A} \times \mathbf{B}|(\mathbf{C} \cdot \hat{\mathbf{n}})$
  - $\hat{\mathbf{n}} = \dfrac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$

- $V = |\mathbf{A} \times \mathbf{B}|(\mathbf{C} \cdot \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

https://i.imgur.com/K6xX1tE.png

7.3 What happens if $\mathrm{a} \times \mathrm{b} = 0$?

7.3.1 Front

7.3.2 Back

We can say that this 2 vector are parallel

7.4 How we can compute cross product and why?

7.4.1 Front

7.4.2 Back

- Compare geometrically the volume of 3 vectors
  - $\det(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$
- $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- $|\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin \theta|$

The result is a **vector**

7.5 Which is the direction of the vector from cross product?

7.5.1 Front

7.5.2 Back

- Result is orthogonal to the plane of the 2 vectors
- Need to use right hand method
- Extend the right hand along 1st vectors
- Move your finger to the other vector
- Your thumb will point to the direction of resulting vector

https://i.imgur.com/movBfMn.png

7.6 What can we say if $|\det(\mathbf{A}, \mathbf{B}, \mathbf{C})| = 0$?

7.6.1 Front

7.6.2 Back

The volume of this parallelepiped with these vectors as edges is 0. This means all three origin vectors lie in a plane.

7.7 What does means $|\mathbf{v}|^2$

7.7.1 Front

Where $\mathbf{v}$ is a vector

7.7.2 Back

$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$

7.8 Determinant - ij-cofactor

7.8.1 Front

What is a ij-cofactor

7.8.2 Back

$A_{ij} = (-1)^{i+j}|A_{ij}|$

7.9 Determinant - ij-entry

7.9.1 Front

What is the **ij-entry**?

7.9.2 Back

a_{ij} is the number in the i-th row and j-th column

7.10 Determinant - ij-minor

7.10.1 Front

What is the **ij-minor**?

7.10.2 Back

$|A_{ij}|$ is the determinant that's left after deleting from $|A|$ the row and column containing a_{ij}

7.11 What is the method of determinant computation?

7.11.1 Front

7.11.2 Back

- It's called the Laplace expansion by cofactors

7.12 How compute a determinant through the Laplace expansion?

7.12.1 Front

Explain how compute a determinant using laplace expansion by cofactors

7.12.2 Back

- Choose a row or a column
- Multiply each entry a_{ij} in that row (or column) by its cofactor A_{ij}
- Add all resulting numbers

Examples

- 1st rows of 3x3 determinant
  - $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
- j-th column of 3x3 determinant
  - $a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}$

7.13 What does mean A_{12}?

7.13.1 Front

7.13.2 Back

This is the 12-cofactor of a determinant, which value is $(-1)^{1+2}|A_{12}|$

$|A_{12}|$ is the ij-minor is a determinant after removing 1st row and 2nd column

7.14 What does mean $|A_{23}|$

7.14.1 Front

7.14.2 Back

This is determinant resulting from removing the 2nd row and 3rd column of the matrix A

7.15 What does mean $|A| = 0$?

7.15.1 Front

7.15.2 Back

- One row or column is all zero
- If two rows or two columns are the same

If A is a matrix that represent 3 vectors 3×3, it means that these 3 vectors are coplanar. This means also that the volume of this parallepiped is 0

7.16 What does mean that $|A|$ is multiplied by c?

7.16.1 Front

7.16.2 Back

- Every element of some row or column is multiply by c

7.17 How we can change the sign of a determinant?

7.17.1 Front

7.17.2 Back

- We interchange two rows or two columns

7.18 When 2 determinants does have the same value?

7.18.1 Front

7.18.2 Back

- If we add to one row (or column) a constant multiple of another row (or column)

https://i.imgur.com/HKZiIZE.png

8 Session 7: Cross Products

Captured On *[2020-02-05 Wed 19:50]*

Source Session 7: Cross Products | Part A: Vectors, Determinants and Planes | 1. Vectors and Matrices | Multivariable Calculus | Mathematics | MIT OpenCourseWare

8.1 Chalkboard

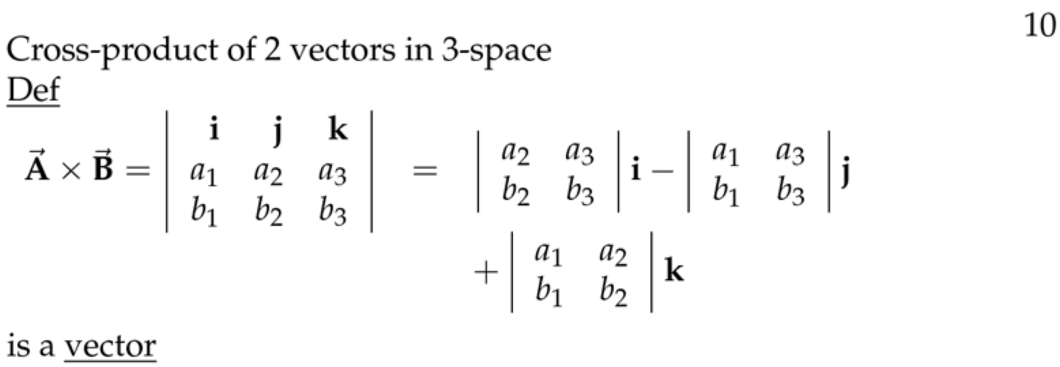
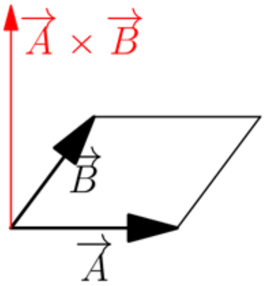


Figure 22: Cross product of 2 vectors in 3-space

Theorem 11



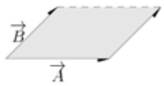
- $|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| =$ area of parallelogram 
- $\text{dir}(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \perp$ to plane of the parallelogram with right hand rule.

Figure 23: Area of parallelogram with cross product

Example 12

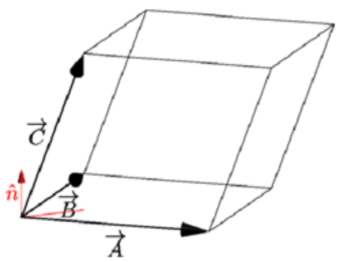
→ Right hand points $\| \vec{\mathbf{A}}$
→ Fingers point $\| \vec{\mathbf{B}}$
→ Thumb points $\| \vec{\mathbf{A}} \times \vec{\mathbf{B}}$

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 1\mathbf{k}$$

Figure 24: Right hand method for direction of cross product

Another look at volume 13



Volume = area(base) height

$= |\vec{\mathbf{B}} \times \vec{\mathbf{C}}|(\vec{\mathbf{A}} \cdot \hat{\mathbf{n}})$

$= |\vec{\mathbf{B}} \times \vec{\mathbf{C}}| \left(\frac{\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}})}{|\vec{\mathbf{B}} \times \vec{\mathbf{C}}|} \right)$

Figure 25: Another look at volume

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$$\det(\vec{\mathbf{A}},\vec{\mathbf{B}},\vec{\mathbf{C}}) \quad = \quad \vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}})$$

$$\downarrow \qquad\qquad \downarrow$$

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + a_3\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} \qquad a_1\cdot\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\left(-\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}\right) + a_3\cdot\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$$

Figure 26: Volume of parallelopiped

1

Last time: cross product

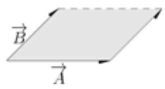
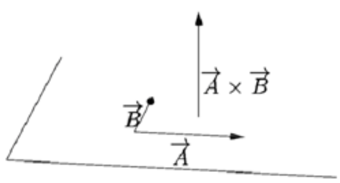
- $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$
- $|\vec{\mathbf{A}}\times\vec{\mathbf{B}}|$ = area of parallelogram 

Figure 27: Review of cross product

2

direction $\perp$ to $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$



Right hand rule.

$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}$ in particular $\vec{\mathbf{A}}\times\vec{\mathbf{A}} = 0$

Figure 28: Inverse direction of cross product

8.2 Can we use the commutivity law with cross product?

8.2.1 Front

$\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$

8.2.2 Back

Not it's not equivalent, but we can say that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

8.3 What we can say of $\mathbf{a} \times \mathbf{a}$

8.3.1 Front

8.3.2 Back

- $\mathbf{a} \times \mathbf{a} = 0$
- There is no area between the same vector

8.4 Can we apply distributive law to cross product?

8.4.1 Front

$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{B}$

8.4.2 Back

Yes

8.5 Can we use associativity law to cross product?

8.5.1 Front

$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

8.5.2 Back

No, you can test with unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$

9 Session 8: Equations of Planes

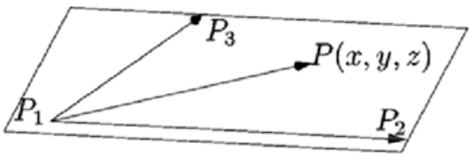
Captured On *[2020-02-05 Wed 19:52]*

Source Session 8: Equations of Planes | Part A: Vectors, Determinants and Planes | 1. Vectors and Matrices | Multivariable Calculus | Mathematics | MIT OpenCourseWare

9.1 Chalkboard

3

- Application: Equation of plane $P_1P_2P_3$



= condition on (x, y, z) telling us whether P is in the plane.

- $\det(\overrightarrow{\mathbf{P_1P_2}}, \overrightarrow{\mathbf{P_1P_3}}, \overrightarrow{\mathbf{P_1P}}) = 0$

Figure 29: Equation of plane

4

Other solution

$$P \text{ is in the plane} \Leftrightarrow \overleftrightarrow{P_1P} \perp \vec{\mathbf{N}} \leftarrow \text{ some vector } \perp \text{ to plane}$$
$$\Leftrightarrow \overrightarrow{\mathbf{P_1P}} \cdot \vec{\mathbf{N}} = 0 \text{ ``normal vector''}$$

How to find $\vec{\mathbf{N}} \perp$ plane? Answer: $\vec{\mathbf{N}} = \overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}}$

(so: $\overrightarrow{\mathbf{P_1P}} \cdot \vec{\mathbf{N}} = \overrightarrow{\mathbf{P_1P}} \cdot (\overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}}) = 0$ triple product = det.)

Figure 30: Another solution equation of plane

9.2 How can we find the equation of the plane containing 3 points?

9.2.1 Front

Points: P_1, P_2, P_3

9.2.2 Back

- Vector $\mathbf{P_1P_2}$ and $\mathbf{P_1P_3}$
- $\mathbf{N} = \mathbf{P_1P_2} \times \mathbf{P_1P_3}$
- $\mathbf{N}$ is perpendicular to the plane
  - $\mathbf{N}$ is normal to the plane

- Vector to any other point in the plane
  - $\mathbf{P_1P} = \langle x - a_1, y - a_2, z - a_3 \rangle$
- $\mathbf{N} \cdot \mathbf{P_1P} = 0$

https://i.imgur.com/ZU1WRar.png Emacs 27.1.50 (Org mode 9.4)